COMPUTATIONAL EFFICIENCY OF PARALLEL APPROXIMATE BRANCH-AND-BOUND ALGORITHMS

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ABSTRACT

A general technique that can be used to solve a wide variety of discrete optimization problems is the branch-and-bound algorithm. We have adapted and extended branch-and-bound algorithms for parallel processing. The computational efficiency of these algorithms depends on the allowance function, the dominance relation, the data structure and the search strategies. Anomalies owing to parallelism may happen and occur frequently when approximations or dominance tests are used. In this paper, sufficient conditions to guarantee that parallelism will not degrade the performance are developed. Necessary conditions for allowing parallelism to have a speedup greater than the number of processors are also presented. Theoretical analysis and simulations show that a best-first search is robust for parallel processing.

KEYWORDS AND PHRASES: Anomalies, approximation, branch-and-bound algorithms, dominance criteria, heuristic search, lower-bound test, NP-hard problems, parallel processing.

1. INTRODUCTION

The search for solutions in a combinatorially large problem space is a major problem in artificial intelligence and operations research. Many practical and theoretical problems such as traveling salesman, warehouse location, job-shop scheduling, knapsack, vertex cover, integer programming, theorem proving, expert systems and game playing can be solved by combinatorial-searches.

Exhaustive search is usually impractical and prohibitively expensive for solving large search problems, especially when the problem is NP-hard. Studies on improving search efficiency is, thus, of considerable importance. Research has been conducted on designing one unified method for a wide variety of problems. The most general technique is the branch-and-bound algorithm [2,6]. This is a partitioning algorithm that decomposes a problem into smaller subproblems and repeatedly decomposes until infeasibility is proved or a solution is found [8,10]. Backtracking, dynamic programming and AND-OR tree search can be viewed as variations of branch-and-bound algorithms.

From the viewpoint of implementation, approximations and parallel processing are two major approaches to enhance the efficiency of branch-and-bound algorithms. Approximations are useful for solving NP-hard problems because a linear reduction in accuracy results in an exponential reduction of the average computational time for solving some NP-hard problems [14]. Properties of serial approximate branch-and-bound algorithms have been studied by Ibaraki [1]. On the other hand, parallel processing is applicable when the problem is solvable in polynomial time, or when the problem is NP-hard, but is solvable in polynomial time on the average [12], or the problem is heuristically solvable in polynomial time (such as game trees). It is not practical to use parallel processing to solve a problem with exponential complexity because an exponential number of processors must be used to solve the problem in polynomial time.

Analytical properties of parallel approximate branch-and-bound algorithms have been rarely studied. From experimental results on vertex-cover, 0-1 knapsack and integer-programming problems, the reduction in computational time of branch-and-bound algorithms with lower-bound tests and a best-first search is approximately linear on the average with respect to the number of processors [13,15]. In general, a k-fold speedup (ratio of the number of iterations in the serial case to that of the parallel case) is expected when k processors are used. However, simulations have shown that the speedup for PABB algorithms using k processors can be (a) less than one--"*detrimental anomaly*" [5,7,11,17]; (b) greater than k--"*acceleration anomaly*" [5,7,17]; or (c) between one and k--"*deceleration anomaly*" [5,7,11,13]. It is desirable to discover conditions that preserve the acceleration anomalies, eliminate the detrimental anomalies and minimize the deceleration anomalies.

The objective of this paper is to study the computational efficiency and anomalous behavior of branch-and-bound algorithms under parallel processing and approximations. These results are helpful for the designers to understand the existence of anomalies and to modify existing algorithms so that detrimental anomalies can be prevented and acceleration anomalies can be enhanced. They can also be used to predict the maximum number of processors needed to attain a near-linear speedup.

2. PARALLEL APPROXIMATE BRANCH-AND-BOUND ALGORITHMS

Many theoretical properties of branch-and-bound algorithms have been developed by Kohler and Steiglitz [6] and Ibaraki [1,2,3]. A brief summary of these properties to solve a minimization problem are given in [15], and only the important properties that will be used in the following sections are stated here.

Branch-and-bound algorithms can be characterized by four constituents: a branching rule, a selection rule, one or more elimination rules and a termination condition. The first two rules are used to decompose problems into simpler subproblems and to appropriately order the search. The last two rules are used to to eliminate generated subproblems that are not better than the ones already known. Appropriately ordering the search and restricting the region searched are the key ideas behind branch-and-bound algorithms.

The way in which P_0 is repeatedly decomposed into smaller subproblems can be represented as a finite rooted tree $\mathbf{B} = (\mathbf{P}, \mathbf{E})$, where $\mathbf{P}$ is a set of subproblems, and $\mathbf{E}$ is a set of edges. The root of the tree is P_0. If a subproblem P_{i_j} is obtained from P_i by decomposition, then $(P_i, P_{i_j}) \in \mathbf{E}$. The *level number* of a node is the number

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of edges leading from the root to this node (the root is at level 0). Let $f(P_i)$ be the value of the best solution obtained by evaluating all the subproblems decomposable from P_i, let P_{i_j} be the j'th subproblem decomposable from P_i, and let k_i be the number of such subproblems (i.e., $k_i = |\{(P_i,x):(P_i,x)\in E\}|$). Then f satisfies:

$$f(P_i) = \min_{j=1,\ldots,k_i} \{f(P_{i_j})\} \qquad (1)$$

Each subproblem is characterized by a lower-bound value that is computed from a lower-bound function g. Let $\mathbf{T}$ be the set of all feasible solutions. The lower-bound function satisfies the following properties:

(a) $g(P_i) \le f(P_i)$ for $P_i \in \mathbf{P}$ (2)
(g is a lower-bound estimate of f)

(b) $g(P_i) = f(P_i)$ for $P_i \in \mathbf{T}$ (3)
(g is exact when P_i is feasible)

(c) $g(P_i) \le g(P_{i_j})$ for $(P_i,P_{i_j}) \in \mathbf{E}$ (4)
(lower bounds always increase)

If a subproblem is a feasible solution with the best objective-function value so far, the solution value becomes the *incumbent* z. The incumbent represents the best solution obtained so far in the process. During the computation, P_i is terminated if:

$$g(P_i) \ge z \qquad (5)$$

The approximate branch-and-bound algorithm is the same as the optimal algorithm except that the lower-bound test is modified to:

$$g(P_i) \ge \frac{z}{1+\epsilon} \qquad \epsilon \ge 0,\ z \ge 0 \qquad (6)$$

where ϵ is an *allowance function* specifying the allowable deviation of a suboptimal value from the exact optimal value. The final incumbent value z_F obtained by the modified lower-bound test deviates from the optimal solution value, z_O, by:

$$\frac{z_F}{1+\epsilon} \le z_O \le z_F \qquad (7)$$

Let $\mathbf{L}$ denotes the lower-bound cutoff test, that is, $P_j \mathbf{L} P_i$ means that P_j is a feasible solution and $f(P_j)/(1+\epsilon) \le g(P_i)$, $\epsilon \ge 0$.

Dominance tests are powerful elimination rules with the use of additional memory space. They are systematically applied in decision problems and in dynamic programming to reduce the complexity of enumeration. Some of the well-known dominance relations are defined for the knapsack, the n-job two-machine flowshop scheduling with minimum mean completion time and the n-job one-machine scheduling with deadlines problems. Dominance tests are rarely used when the problem is inherently intractable.

A dominance relation is a binary relation such that $P_i \mathbf{D} P_j$ (or $(P_i,P_j) \in \mathbf{D}$) implies P_i dominates P_j [1,6]. This means that the subtree rooted at P_i contains a solution node with a value no more than the minimum solution value of the subtree rooted at P_j. Thus if P_i and P_j are generated and $P_i \mathbf{D} P_j$, then P_j can be terminated. A dominance relation satisfies the following conditions [4]:

(a) that $P_i \mathbf{D} P_j$ implies $f(P_i) \le f(P_j)$ and that P_i is not a proper descendant of P_j; (8)

(b) that $\mathbf{D}$ is a partial ordering (reflexive, antisymmetric and transitive); (9)

(c) that $P_i \mathbf{D} P_j$ and $P_i \ne P_j$ imply that for any proper descendant $P_{j'}$ of P_j, there exists a descendant $P_{i'}$ (including P_i) of P_i such that $P_{i'} \mathbf{D} P_{j'}$. (10)

To implement dominance tests, only the set of nodes that have been generated and have not been dominated so far have to be stored. These nodes are called the *current dominating nodes*. In general, the set of active nodes is not sufficient to determine the set of current dominating nodes because $P_i \mathbf{D} P_j$ does not imply that there exists a proper descendant $P_{i'}$ of P_i such that $P_{i'} \mathbf{D} P_j$ (Eq. 10).

Ibaraki mapped breadth-first, depth-first and best-first searches into a general form called *heuristic searches* [2]. A heuristic function is defined that governs the order in which subproblems are selected and decomposed. The algorithm always decomposes the subproblem with the minimum heuristic value. In a best-first search, the lower-bound values define the order of expansion. Therefore, the lower-bound function can be taken as the heuristic function. In a breadth-first search, subproblems with the minimum level numbers are expanded first. The level number can, thus, be taken as the heuristic function. Lastly, in a depth-first search, subproblems with the maximum level numbers are expanded first. The negation of the level number can be taken as the heuristic function. If $\mathbf{U}$ is the current list of active subproblems in the process of expansion and h is the heuristic function, the search function for a subproblem to expand in a serial branch-and-bound algorithm is:

$$s_s(\mathbf{U}) = \{P_i \mid h(P_i) = \min_{P_j \in U} h(P_j)\} \qquad (11)$$

Branch-and-bound algorithms have inherent parallelism. Each of the four rules of serial branch-and-bound algorithms can be implemented by parallel processing.

(a) *Parallel selection of subproblems*: In the parallel case, a set of subproblems less than or equal in size to the number of processors have to be selected for decomposition in each iteration. The selection problem is especially critical under a best-first search because a set of subproblems with the minimum lower bounds must be selected. The selection function (Eq. 11) becomes,

$$s_P(\mathbf{U}) = \begin{cases} \underset{\substack{\text{where } h(P_i) < h(P_j),\ P_i,\ P_j \in U \\ i \in \{i_1,\ldots,i_k\},\ j \notin \{i_1,\ldots,i_k\}}}{\{P_{i_1},\ldots,P_{i_k}\}} & \text{if } |U| > k \\ \mathbf{U} & \text{if } |U| \le k \end{cases} \qquad (12)$$

where k is the number of processors. This returns the set of k subproblems with the minimum heuristic values from U.

(b) *Parallel branch*: The subproblems assigned to the processors can be decomposed in parallel. In order for the processors to be well utilized, the number of active subproblems should be greater than k.

(c) *Parallel termination test*: Multiple infeasible nodes can be eliminated in each iteration. Further, multiple feasible solutions may be generated, and the incumbent has to be updated in parallel.

(d) *Parallel elimination test*: The lower-bound test (Eq's 5 or 6) can be sped up by comparing lower bounds of multiple subproblems with the incumbent. The elimination of subproblems by dominance tests can also be carried out in parallel. However, the bounding function and the dominance test are problem-dependent, and software implementation is more flexible.

3. ANOMALIES ON PARALLEL BRANCH-AND-BOUND ALGORITHMS

In this section, some anomalies on parallel approximate branch-and-bound algorithms are illustrated. The parallel computation model for studying the anomalies consists of a set of processors connected by a network. There can be a single subproblem list for all the processors, or there are multiple lists, one for each processor. The distinction lies in the memory configuration. When all the processors are connected to a centralized memory, the subproblem list is global to all the processors. When each processor has a private memory, only the local subproblem list can be accessed. The workload is balanced through the network. It is assumed that the processors operate synchronously in executing the steps of the branch-and-bound algorithm.

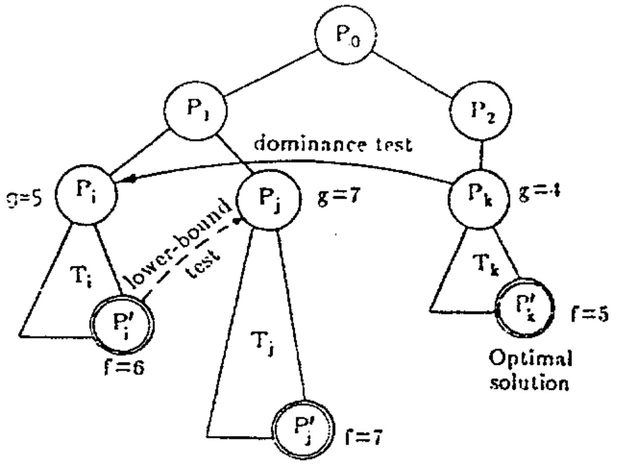
Let $T^c(k,\epsilon)$ and $T^d(k,\epsilon)$ denote the number of iterations required for expanding a branch-and-bound tree using centralized and k subproblem lists respectively, where k is the total number of processors used, and ϵ is the allowance function. Since the subproblems are decomposed synchronously, the number of iterations is measured by the number of times that subproblems are decomposed in each processor.

For the branch-and-bound tree in Figure 1a, the order of nodes expanded using a depth-first search on a single processor is P_0, P_1, P_i, all the nodes in T_i that result in a feasible solution $P_{i'}$ with value 6, P_k, and all the nodes in T_k that result in the optimal solution of $f(P_{k'})=5$. P_j is terminated by the lower-bound test using $P_{i'}$. In contrast, when two processors are used, P_1 and P_2 are expanded concurrently. After the expansion, P_i is terminated as a result of dominance by P_k. Since T_i is terminated, P_j and T_j must be expanded. If subtree T_j is large, the combined time of expanding T_i and T_k using one processor can be smaller than the combined time of expanding T_j and T_k using two processors. This anomaly will be investigated in Section 5.2.

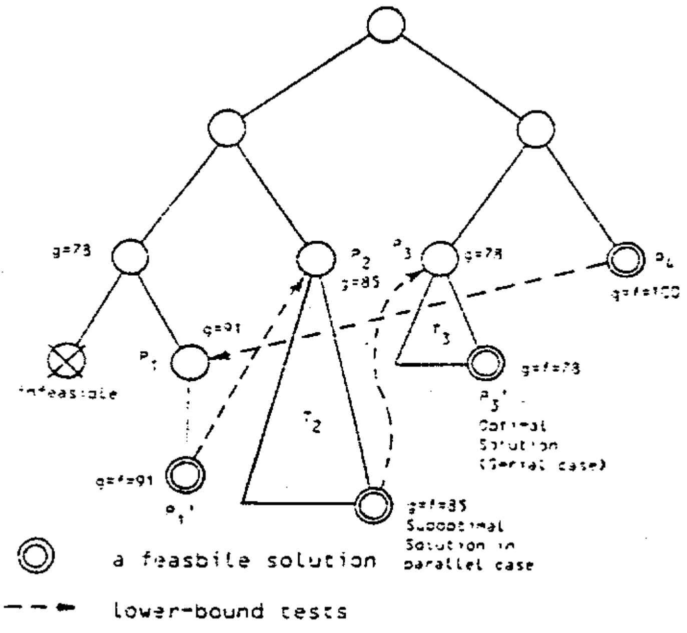
An example illustrating a detrimental anomaly when approximations are allowed is shown in Figure 1b. In a serial depth-first search, subtree T_2 is terminated owing to the lower-bound test of P_1': $f(P_1')/(1+\epsilon) \leq g(P_2)$ where $\epsilon=0.1$. In a parallel depth-first search with two processors, a feasible solution P_4 is found in the second iteration, and nodes P_1 and P_1 are terminated owing to the lower-bound test of P_4. Consequently, subtree T_2 has to be expanded which will eventually terminate subtree T_3. If the size of T_2 is much larger than that of T_3, the time it takes to expand T_2 using two processors will be longer than the time it takes to expand T_3 using one processor. Detrimental anomalies with approximations are discussed in Section 5.3.

Figure 2a shows an acceleration anomaly with dominance and lower-bound tests under a depth-first search. In using a single processor, T_i, T_i and T_k are expanded. When two processors are used, P_2 is expanded in the second iteration, and nodes P_i and P_j are terminated. Therefore, only T_k is expanded in the parallel case. A similar example on lower-bound tests with approximations is shown in Figure 2b. Acceleration anomalies will be discussed in Section 6.

Other anomalous examples can be created for breadth-first and best-first searches. However, the important consideration here is not in knowing that anomalies exist, but in understanding why these anomalies occur. Our objective is to find the sufficient conditions to ensure that $T^c(k,\epsilon) \leq T^c(1,\epsilon)$ and $T^d(k,\epsilon) \leq T^d(1,\epsilon)$, as well as the necessary conditions for $T^c(k,\epsilon) \leq T^c(1,\epsilon)/k$ and



(a) with lower-bound and dominance tests ($\epsilon=0$).



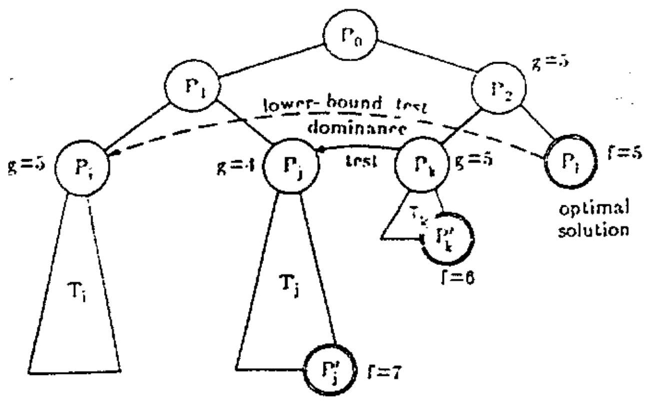
(b) with lower-bound tests only ($\epsilon>0$).

Figure 1. Examples of detrimental anomalies under a depth-first search.

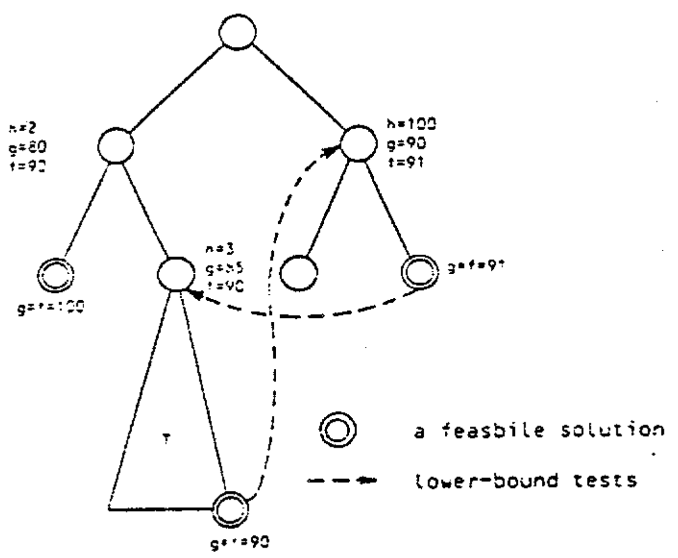
$T^d(k,\epsilon) \leq T^d(1,\epsilon)/k$. Note that when there is one processor, only one memory system is used, and $T^c(1,\epsilon)=T^d(1,\epsilon)$. The necessary conditions to eliminate detrimental anomalies are not evaluated because they are problem dependent. A condition necessary for avoiding detrimental anomalies depends on the sequence of nodes expanded and the size of the resulting subtrees. There are many possible combinations, and to enumerate them for a given problem is difficult. Furthermore, the necessary conditions developed for one problem cannot be generalized to other problems. These are also the reasons for not evaluating the sufficient conditions to preserve acceleration anomalies.

4. GENERALIZED HEURISTIC SEARCHES

Recall in Sections 2 that the selection function uses heuristic values to define the order of node expansions.



(a) with lower-bound and dominance tests ($\epsilon=0$).



(b) with lower-bound tests only ($\epsilon>0$).

Figure 2.
Examples of acceleration anomalies under a depth-first search.

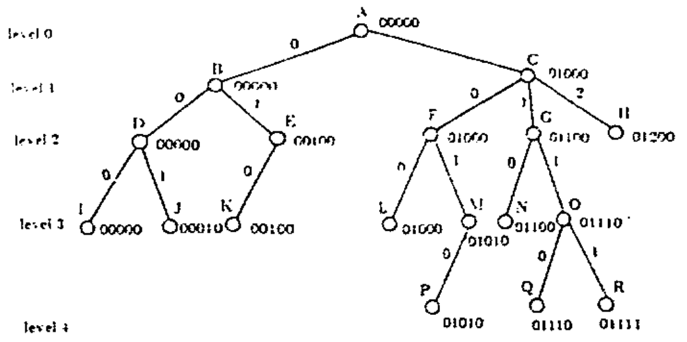


Figure 3.
The path numbers of a tree.

In this section, we show that detrimental anomalies are caused by the ambiguity in the selection rule. A generalized heuristic search is proposed to eliminate detrimental anomalies in a single subproblem list.

Consider the serial depth-first search. The subproblems are maintained in a last-in-first-out list, and the subproblem with the maximum level number is expanded first. When multiple subproblems have identical level numbers (heuristic values), the node chosen depends on the order of insertion into the stack. Suppose the rightmost son of a parent node is always expanded and inserted first. Then the leftmost son will be inserted last and expanded first in the next iteration.

In a parallel depth-first search, the mere extension of the serial algorithm may cause an anomalous behavior. For example, the order of expansion in a serial depth-first search for the tree in Figure 3 is A, B, D, I, J, E, etc. When two processors are used, nodes B and C are decomposed to nodes D, E, F, G and H in the second iteration. Since these nodes have identical level numbers, the search function can choose any two of these nodes for expansion in the next iteration. Suppose the nodes are inserted in the order E, D, H, G and F. Then nodes F and G will be selected and expanded in the third iteration. This may cause an unexpected behavior as compared to the serial case and is exactly the reason for the anomalies reported in [7].

To solve this problem, we must define distinct heuristic values for the nodes so that there is no ambiguity on the nodes to be chosen by the selection function. In this paper, a path number is used to uniquely identify a node. The *path number* of a node in a tree is a sequence of d+1 integers that represent the path from the root to this node where d is the maximum level number of the tree. The path number $E = e_0e_1e_2...e_d$ is defined recursively as follows. The root P_0 exists at level 0 and has a path number of 000...0. A node P_{i_j} on level ℓ that is the j'th son (counting from the left) of P_i with path number $E_{P_i} = e_0e_1...e_{\ell-1}000...$ has path number $E_{P_{i_j}} = e_0e_1...e_{\ell-1}j00....$ As an example, the path numbers for all the nodes in the tree of Figure 3 are shown.

To compare path numbers, the relations ">" and "=" must be defined. A path number $E_1 = e_1^1e_2^1 \cdots$ is less than another path number $E_2 = e_1^2e_2^2 \cdots$ ($E_1 < E_2$) if there exists $0 \le j \le d$ such that $e_i^1 = e_i^2$, $0 \le i < j$, and $e_j^1 < e_j^2$. The path numbers are equal if $e_i^1 = e_i^2$ for $0 \le i \le d$. For example, the path number 01000 is less than 01010. Note that nodes can have equal path numbers if they have the ancestor-descendant relationship. Since these nodes never coexist simultaneously in the list of active subproblems of a branch-and-bound algorithm, the subproblems in the active list always have distinct path numbers.

The path number is now included in the heuristic function. The primary key is still the lower-bound value or the level number. The secondary or ternary key is the path number and is used to break ties in the primary key.

$$h(P_i) = \begin{cases} \text{(level number, path number)} \\ \qquad \text{for breadth-first search} \\ \text{(path number)} \\ \qquad \text{for depth-first search} \\ \text{(lower bound, level number, path number) or} \\ \text{(lower bound, }-\text{level number, path number)} \\ \qquad \text{for best-first search} \end{cases} \qquad (13)$$

For a best-first search, two alternatives are defined that search in a breadth-first or depth-first fashion for nodes with identical lower bounds. The heuristic functions defined above belong to a general class of heuristic functions that satisfy the following properties:

(a) $h(P_i) \neq h(P_j)$ if $P_i \neq P_j$ $P_i, P_j \in \mathbf{P}$ (14)
All heuristic values in the active list are distinct.

(b) $h(P_i) \leq h(P_j)$ if P_j is a descendant of P_i (15)
(heuristic values do not decrease)

In general, the results developed in the following sections are applicable to any heuristic function that satisfy Eq's 14 and 15.

6. SUFFICIENT CONDITIONS TO ELIMINATE DETRIMENTAL ANOMALIES IN A SINGLE SUBPROBLEM LIST

6.1 Parallel Branch-and-Bound with Lower-Bound Tests Only

In this section, we show that any heuristic search with a heuristic function that satisfies Eq's 14 and 15 can guarantee that $T^c(k,\epsilon) \leq T^c(1,\epsilon)$ when only lower-bound tests with $\epsilon = 0$ are used in a system with a centralized subproblem list. First, we define a *basic node* as the node with the smallest heuristic value in each iteration. All the nodes expanded in a serial algorithm are basic nodes. Let Φ^1 and Φ^k be the sets of nodes in the branch-and-bound tree expanded using one and k processors respectively. To show that $T^c(k,0) \leq T^c(1,0)$, it is necessary to prove: (a) that at least one node that belongs to Φ^1 (the basic node) is expanded in each iteration of the parallel search; and (b) that once all the nodes in Φ^1 are expanded or terminated, the parallel heuristic search must terminate. The proof requires the following property on basic nodes.

Lemma 1: Let P_i be a basic node, then for any node P_j such that $h(P_j) < h(P_i)$, P_j must be either expanded or terminated when P_i is expanded.

Proof: Suppose in the current active list, $P_i \in U$ is a basic node. Assume that there exists a node P_j such that $h(P_j) < h(P_i)$ and P_j has not been expanded or terminated when P_i is expanded. Since P_i has the minimum heuristic value among the set of currently active nodes U, P_j must not be active at that time. That is, P_j is a descendant of some node P_k, $P_k \in U$, and $h(P_i) < h(P_k)$. By Eq. 15, $h(P_i) < h(P_k) \leq h(P_j)$ which contradicts the assumption that $h(P_j) < h(P_i)$. □

The following theorem proves that any selection function with properties given in Eq's 14 and 15 are sufficient to eliminate the detrimental anomalies.

Theorem 1: Let $\epsilon = 0$, $D = I$ (identity relation), i.e., an exact optimal solution is sought and the dominance tests are not active. $T^c(k,0) \leq T^c(1,0)$ holds for parallel heuristic searches of a single optimal solution in a centralized list using any heuristic function with properties defined in Eq's 14 and 15.

Proof: The proof is by contradiction. Suppose there exists a basic node P_{i_1} such that $P_{i_1} \notin \Phi^1$ and $P_{i_1} \in \Phi^k$ (Figure 4). This means that P_{i_1} is terminated by a lower-bound test in the serial case, and there must exist a feasible solution $P_{i_2} \in \Phi^1$ such that $f(P_{i_2}) \leq g(P_{i_1})$ and P_{i_2} has not been obtained when P_{i_1} is expanded in the parallel case. It implies that a proper ancestor $P_{i_3} \in \Phi^1$ of P_{i_2} exists such that $h(P_{i_3}) < h(P_{i_1})$, and P_{i_2} is obtained before P_{i_1} and terminates P_{i_1}. Since P_{i_1} is a basic node in the parallel search, $h(P_{i_3}) < h(P_{i_1})$ and P_{i_3} has not been expanded when P_{i_1} is expanded in the parallel case, P_{i_3} must be terminated according to Lemma 1. For the parallel search, there must exist a feasible solution $P_{i_4} \in \Phi^k$ such that $f(P_{i_4}) \leq g(P_{i_3})$, and P_{i_4} has not been obtained when P_{i_3} is expanded in the serial case. Two cases are possible:

First, P_{i_4} is not generated when P_{i_3} is expanded in the serial case, i.e., a proper ancestor $P_{i_5} \in \Phi^k$ of P_{i_4} exists such that $h(P_{i_5}) > h(P_{i_3})$. According to the properties of lower-bound functions (Eq's 2, 3 and 4), we have $f(P_{i_4}) \leq g(P_{i_3}) \leq f(P_{i_3}) \leq f(P_{i_2}) \leq g(P_{i_1})$. Moreover, in the parallel case, P_{i_4} has been obtained when P_{i_3} is generated because P_{i_3} is terminated by P_{i_4}. Since $h(P_{i_3}) < h(P_{i_1})$, P_{i_3} must be generated before P_{i_1} by Lemma 1. Consequently, P_{i_4} is available when P_{i_1} is expanded. Hence, P_{i_1} has to be terminated in the parallel algorithm which contradicts the assumption that $P_{i_1} \in \Phi^k$.

Second, $h(P_{i_5}) < h(P_{i_3})$, and P_{i_5}, as well as its descendant P_{i_4}, have been terminated in the serial case and not in the parallel case. We can then apply the above argument again to P_{i_5}, and eventually obtain a sequence of nodes $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$ as depicted in Figure 4. There are three possibilities:

(a) The first node P_{i_m} occurs in the serial case (Figure 4a). Since $h(P_{i_{k+2}}) < h(P_{i_k})$, otherwise P_{i_k} cannot be terminated by $P_{i_{k+1}}$ in the serial case, and since $h(P_{i_{k+4}}) < h(P_{i_{k+2}})$ by the same argument as $h(P_{i_5}) < h(P_{i_3})$, we have $h(P_{i_{k+4}}) < h(P_{i_k})$. Repeating this, we get $h(P_{i_m}) < h(P_{i_1})$. By Lemma 1, P_{i_m} must be expanded in the parallel case and terminates P_{i_1} which contradicts that $P_{i_1} \in \Phi^k$.

(b) The first node P_{i_m} occurs in the parallel case (Figure 4b). Since $P_{i_{m-1}}$ is a feasible solution, we have: $f(P_{i_{m-1}}) \leq g(P_{i_{m-2}}) \leq f(P_{i_{m-2}}) \leq g(P_{i_{m-3}}) \leq \cdots \leq f(P_{i_1}) \leq g(P_{i_1})$. Similar to the argument for P_{i_4} discussed in the first case, we can explain that $P_{i_{m-1}}$ has been obtained when P_{i_1} is selected. Therefore, P_{i_1} is terminated in the parallel case which contradicts that $P_{i_1} \in \Phi^k$.

(c) There is a cycle of cutoffs such that $P_{i_{m-1}} \mathbf{L} P_{i_{m-2}}$, $P_{i_{m-3}} \mathbf{L} P_{i_{m-4}}, \ldots, P_{i_{k-1}} \mathbf{L} P_{i_k}$ and $P_{i_{k-1}} \mathbf{L} P_{i_m}$ where $\mathbf{L}$ denotes a lower-bound cutoff test (Figure 4c). By transitivity, we have $f(P_{i_m}) \leq f(P_{i_{m-1}}) \leq f(P_{i_{k-1}}) \leq f(P_{i_m})$ which implies that $f(P_{i_m}) = f(P_{i_{m-1}}) = f(P_{i_{k-1}})$. The heuristic value of all the nodes of the cycle are less than $h(P_{i_1})$, so a feasible solution has been obtained

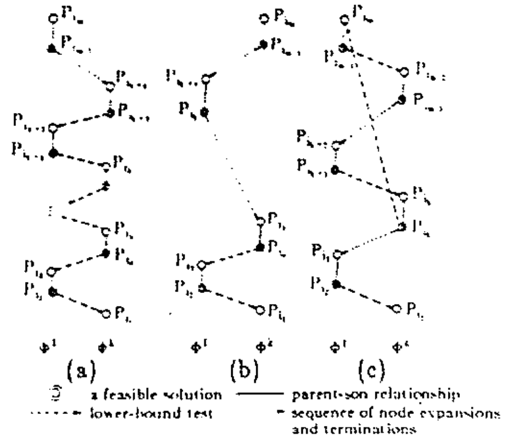


Figure 4. Proof of Theorem 1.

before P_{i_j} is selected. Thus P_{i_j} must be terminated in the parallel case.

So far we have proved that at least one node of Φ^1 is expanded in each iteration of a parallel heuristic search. It also implies that after all the nodes of Φ^1 are expanded or terminated in the parallel heuristic search, no active node remains. □

The above theorem shows that detrimental anomalies can be avoided for depth-first, breadth-first and best-first searches with $\epsilon=0$ and $\mathbf{D}=\mathbf{I}$ by augmenting the heuristic values with the path numbers. As a special case, for a best-first search in which all the lower bounds are distinct, the nodes with the smallest lower bounds are always selected through the priority queue. Hence the path numbers do not have to be used, and no detrimental anomaly will occur. A more general condition will be shown in Section 8.

5.2 Parallel Branch-and-Bound with Lower-Bound and Dominance Tests

When lower-bound and dominance tests are used together, it is important to note that dominance tests are not transitive with lower-bound tests. That is, $P_i \mathbf{L} P_j$ and $P_j \mathbf{D} P_k$ do not imply $P_i \mathbf{L} P_k$. Similarly, $P_i \mathbf{D} P_j$ and $P_j \mathbf{L} P_k$ do not imply $P_i \mathbf{D} P_k$. In both cases, only $f(P_i) \leq f(P_k)$ can be deduced. Recall that when $P_i \mathbf{D} P_k$, it implies $f(P_i) \leq f(P_j)$. (Eq. 8) The converse is not true because otherwise the number of active nodes can always be reduced to one.

Dominance tests together with lower-bound tests are more apt to enhance the parallel branch-and-bound algorithms than if lower-bound tests were used alone; however, detrimental anomalies are more difficult to prevent. The sufficient conditions developed here to ensure that $T^c(k,0) \leq T^c(1,0)$ are stronger than those conditions developed in the last section. These conditions are defined for special classes of heuristic functions and dominance relations. A heuristic function h is said to be *consistent* with the lower-bound function g if $h(P_i) < h(P_j)$ implies that $g(P_i) \leq g(P_j)$ for all $P_i, P_j \in \mathbf{P}$. A dominance relation $\mathbf{D}$ is said to *consistent* with the heuristic function h if $P_i \mathbf{D} P_j$ implies that $h(P_i) < h(P_j)$ for all $P_i, P_j \in \mathbf{P}$. The following theorem proves that these conditions are sufficient.

Theorem 2: Let $\epsilon = 0$, $\mathbf{D} \neq \mathbf{I}$. $T^c(k,0) \leq T^c(1,0)$ holds for heuristic searches that satisfy Eq's 14 and 15 and the following conditions: (a) heuristic function h is consistent with lower-bound function g, and (b) Dominance relation $\mathbf{D}$ is consistent with h.

Due to space limitation, the proofs of Theorem 2 and the following theorems are omitted. All of the proofs can be found in [9].

One possible way in which the two consistency requirements in Theorem 2 can be satisfied would be: (a) to use a best-first search, (b) to require that all the lower bounds are distinct or to use path numbers to break the tie for identical lower bounds, and (c) to use a dominance relation that is *consistent* with g, i.e., $P_i \mathbf{D} P_j$ implies that (1) $g(P_i) < g(P_j)$ or (2) $g(P_i) = g(P_j)$ and $h(P_i) < h(P_j)$. The first requirement implies that the heuristic function is consistent with the lower-bound function. The last two requirements are true in order for the dominance relation to be consistent with the heuristic function.

The requirement on the consistency of the dominance relation with respect to g is satisfied in many practical problems. For instance, for the 0-1 knapsack problem, the upper bound of the maximization problem can be computed by arranging the objects in decreasing profit-to-weight ratios and allocating as many objects as possible into the knapsack while allowing fractional allocation of last object packed. A dominance relation is consistent with the upper bound function for the knapsack problem if it is defined as follows. For P_i and P_j defined on a given subset of objects, $P_i \mathbf{D} P_j$ if (1) the profit of P_i is larger than that of P_j and the total weight of P_i is less than that of P_j; or (2) the profit and weight of P_i are equal to the corresponding profit and weight of P_j and the path number of P_i is less than that of P_j. Likewise, the hybrid dynamic-programming/branch-and-bound algorithm for solving the traveling-salesman problem has a dominance relation that is consistent with g. So are the dominance relations for the shortest-path problem [3], the n-job two-machine mean-finishing-time flowshop problem and the n-job one-machine scheduling problem with deadlines [SAH76]. Note that the example in Figure 1a does not satisfy the requirement that $\mathbf{D}$ is consistent with h.

5.3 Parallel Branch-and-Bound Algorithms with Approximations

When parallel approximate branch-and-bound algorithms are considered, Theorems 1 and 2 are no longer valid (see the example in Figure 1b). The reason for the anomaly is that the lower-bound tests under approximation, L, are not transitive. That is, $P_i \mathbf{L} P_j$ and $P_j \mathbf{L} P_k$ do not imply $P_i \mathbf{L} P_k$, since $f(P_i)/(1+\epsilon) \leq g(P_j)$ and $f(P_j)/(1+\epsilon) \leq g(P_k)$ implies $f(P_i)/(1+\epsilon)^2 \leq g(P_k)$ rather than $f(P_i)/(1+\epsilon) \leq g(P_k)$. Somewhat surprisingly, it is possible that Φ^1 and Φ^k are almost disjoint, and most of the nodes in Φ^1 are not expanded in the parallel case. Detrimental anomalies can be avoided for a best-first or a breadth-first search under the same conditions as stated in Theorem 2.

Theorem 3: $T^c_?(k,\epsilon) \leq T^c(1,\epsilon)$, $\epsilon>0$, holds for parallel best-first searches for a single optimal solution when: (a) heuristic function, h, satisfies Eq's 14 and 15; (b) dominance relation, $\mathbf{D}$, is consistent with h.

Condition (a) in the above theorem is satisfied if all the lower-bound values are distinct. Otherwise, the identical lower bounds must be augmented by the path numbers to break the tie. For depth-first searches, the conditions of Theorem 3 are not sufficient, and it is hard to avoid the detrimental anomalies in this case.

6. NECESSARY CONDITIONS FOR ACCELERATION ANOMALIES IN A SINGLE SUBPROBLEM LIST

In this section, the necessary conditions for $T^c(k,0) < T^c(1,0)/k$ are developed. One of these conditions is based on the complement of the special class of dominance relations defined in Section 5.2. A dominance relation, $\mathbf{D}$, is said to be *inconsistent* with h if two nodes P_i and P_j exist such that $P_i \mathbf{D} P_j$ and $h(P_i) > h(P_j)$. Another condition is based on the complete consistency of heuristic functions. A heuristic function, h, is said to be *not completely consistent* with g if there exist two nodes P_i and P_j such that $h(P_i) > h(P_j)$ and $g(P_i) \leq g(P_j)$. Note that this is not exactly the complement of the consistency of heuristic functions defined in the last section. To illustrate the difference, if $g(P_i)=g(P_j)$ is allowed, then the heuristic function for a best-first search is consistent, but not completely consistent, with the lower-bound function.

Theorem 4: Let $\epsilon = 0$ and assume that a single optimal solution is sought. The necessary condition for $T^c(k,0) < T^c(1,0)/k$ is either that (a) the heuristic function is not completely consistent with g, or (b) the dominance relation is inconsistent with h.

For instance, the example in Figure 2a satisfies the conditions of Theorem 4. According to Theorems 2 and 4, we can conclude that if dominance tests are active, both detrimental and acceleration anomalies exist when a breadth-first or a depth-first search is used. However, it is possible that acceleration anomalies exist and detrimental anomalies can be prevented for best-first searches (owing to the difference between consistency and complete consistency). It is important to note that the conditions in Theorem 4 are not necessary when approximate solutions are sought. i.e., acceleration anomalies may occur even though h is complete consistent with g and the dominance tests are consistent with h. In summary, acceleration anomalies may occur in one of the followings cases: (a) when using a breath-first or a depth-first search; (b) when some nodes have identical lower bound; (c) when the dominance relation is inconsistent with the heuristic function; (d) when a suboptimal solution is sought; or (e) when using multiple lists of subproblems (discussed in the next section).

7. MULTIPLE SUBPROBLEM LISTS

When there are multiple subproblem lists, one for each processor, a node is selected from each local list with the minimum heuristic value for decomposition. This node may not belong to the global set of active nodes with the minimum heuristic values, but the node with the minimum heuristic value will always be expanded by one processor as long as the order of insertion into a subproblem list is the same for the serial and the parallel searches. Since it is easy to maintain the incumbent in a global data register, the behavior of multiple lists is analogous to that of a centralized list. Hence, when dominance tests are inactive ($D = I$), no detrimental anomaly occurs. However, the performance of using multiple lists is usually worse than that of a single subproblem list [18].

When dominance tests are active, these tests can be restricted to the local set of current dominating nodes or they can be performed on all the current dominating nodes. If the dominance tests are carried out on all the current dominating nodes, it is necessary to broadcast newly generated nodes to all the processors and to balance the sizes of the multiple sets of current dominating nodes at the end of each iteration. The behavior is similar to that of a centralized list. On the other hand, if the dominance tests are performed on the local set of current dominating nodes, then it is possible that $P_i D P_j$ and P_j is not terminated because the two nodes are stored in different processors. As a result, both detrimental and acceleration anomalies may occur. For instance, a detrimental anomaly may happen because a subtree rooted at P_j is pruned in the serial case and is not pruned in the parallel case. However, if P_j has a feasible solution $P_{j'}$ such that $P_{j'} L P_k$ and $P_{j'}$ and P_k are located in the same processor, then the subtree rooted at P_k will be pruned in the parallel case because $P_i D P_j$ is inactive and not in the serial case because P_j is terminated by $P_i D P_j$. This may introduce an acceleration anomaly. It is found that anomalies happen more frequently when dominance tests are performed on local current dominating nodes.

8. ROBUSTNESS OF PARALLEL BEST-FIRST SEARCHES

The results in the previous sections are useful for preventing detrimental anomalies and preserving acceleration anomalies. However, they do not apply when deceleration anomalies are concerned. Deceleration anomalies are usually unavoidable and diminishing returns occur when the number of processors is large. So far, all known results showed that for a depth-first search, a near-linear speedup holds only for a small number of processors. In addition, the generalized heuristic search mentioned in Section 4 cannot guarantee that $T(k_2, \epsilon) \leq T(k_1, \epsilon)$, $k_1 < k_2$, even with $\epsilon = 0$ and inactive dominance tests. The results in this section indicate that a best-first search with lower-bound tests only has the best behavior as far as deceleration anomalies and relative performance are concerned. Since the performance is not affected by using single or multiple subproblem lists, the superscript in T is dropped.

Theorem 5: For a parallel best-first search with k processors, $\epsilon=0$, $D=I$ and $g(P_i) \neq f^*$ if P_i is not an optimal-solution node,

$$\frac{T(1,0)}{k} \leq T(k,0) \leq \frac{T(1,0)}{k} + \frac{k-1}{k}\hat{\ell} \qquad (16)$$

where $\hat{\ell}$ is the maximum number of levels of the branch-and-bound tree to be searched.

In most applications, the height of the branch-and-bound tree is a polynomial function of the problem size. Hence, the second term on the right hand side of Eq. 16 is much smaller than the first term, and a near-linear speedup holds for best-first searches in a considerable range of the number of processors. As an example, if $\hat{\ell} = 50$, $T(1,0) = 10^6$ (for a typical traveling-salesman problem), and $k=1000$, then $1000 \leq T(1000,0) \leq 1049$. This means that almost linear speedup can be attained with one thousand processors.

From Theorem 5, it is also easy to determinate the maximum number of processors that must be used to have a near-linear speedup. Assume that the speedup required is $T(1,0)/(\eta k) \geq T(k,0)$, $0 < \eta < 1$. From Eq. 16,

$$T(k,0) \leq \frac{T(1,0)}{k} + \frac{k-1}{k}\hat{\ell} \leq \frac{T(1,0)}{\eta k}$$

This results in:

$$k \leq \frac{1-\eta}{\eta \hat{\ell}} T(1,0) + 1 \qquad (17)$$

For instance, if $\eta=0.9$, $\hat{\ell}=50$, and $T(1,0)=10^6$, then $k \leq 2223$. That is, a minimum of 0.9 speedup is obtained if 2223 or less processors are used.

Lastly, we can obtain the sufficient condition for assuring the monotonic increase in computational efficiency of parallel branch-and-bound algorithms with a best-first search. If the assumptions of Theorem 5 are satisfied, $T(k_2,0) \leq T(k_1,0)$, $1 < k_1 < k_2$, when:

$$T(1,0) \geq \frac{k_1(k_2-1)\hat{\ell}}{k_2-k_1} \qquad (18)$$

For $k_2 > k_1$, since $k_2^2 > k_1(k_2-1) \geq k_1(k_2-1)/(k_2-k_1)$, Eq. 18 is true if $T(1,0) \geq k_2^2 \hat{\ell}$. In other words, there is always a monotonic increase in performance for all $1 \leq k_1 < k_2 \leq \sqrt{T(1,0)/\hat{\ell}}$. For example, for $\hat{\ell}=50$, $T(1,0)=10^6$, there will not be any risk of detrimental anomalies for any combinations of $1 \leq k_1 < k_2 \leq 141$.

Theorem 5 assumes that $g(P_i) \neq f^*$ unless P_i is an optimal-solution node. When this assumption is relaxed, acceleration anomalies may occur. Simulation results on 0-1 knapsack and vertex-cover problems revealed that the speedup of a parallel best-first search is much better than that of a parallel depth-first search when the number of processors is large [9].

9. CONCLUSION

Branch-and-bound algorithms can be used to solve a wide variety of combinatorial-search problems.

Anomalies may occur when parallelism and approximation are applied owing to the ambiguities of heuristic functions and inconsistencies between the selection rule and the elimination rules. A method of using the path numbers is proposed in this paper to assign a distinct heuristic value to each node and to uniquely order the search in the branch-and-bound tree. Other conditions on consistencies of dominance relations and heuristic functions are also developed. A summary of the results proved in this paper are shown in Table 1. Note that these results are developed for finding a single solution. If all the solutions are sought, the corresponding conditions can be investigated similarly.

A best-first search is found to be a good search strategy that can guarantee little degradation in performance with increasing degrees of parallelism. It is more suitable for parallel processing than breadth-first or depth-first searches due to the large number of active subproblems. More general results on the relative performance between k_1 and k_2 processors also show that best-first searches are more robust as far as deceleration anomalies are concerned. A near-linear speedup with respect to the number of processors has been observed for parallel best-first searches [13,15]. Owing to the large number of active subproblems in a best-first search, the effective mapping of active subproblems onto the secondary storage is an important problem. It was found that a direct implementation of the algorithm on a memory hierarchy resulted in an intolerable amount of disk accesses. A modified best-first search that can overlap all the secondary-storage accesses with computations has been proposed and studied [16,18].

Table 1. Summary of results for the elimination of detrimental anomalies and the preservation of acceleration anomalies in parallel branch-and-bound algorithms.

Allowance function	Dominance relation	Subproblem lists	Search strategies	Suff. cond. to eliminate Detrimental Anomaly	Necessary cond. for Acceleration Anomaly
$\epsilon=0$	$D=I$	Single	all	I	IV
		multiple		no anomaly	
	$D\neq I$	single	all	I, II, III	IV or V
		multiple		anomaly*	exists
$\epsilon>0$	$D=I$	single or multiple	breadth-first or best-first	I	exists
			depth-first	anomaly	
	$D\neq I$	single	breadth-first or depth-first	anomaly	
			best-first	I, II, III	
		multiple	all	anomaly*	

Conditions:

I:	heuristic function satisfies Eq's 14 and 15.
II:	h is consistent with g, i.e., $h(P_i)<h(P_j)$ implies $g(P_i)\leq g(P_j)$.
III:	D is consistent with h, i.e., $P_i D P_j$ implies $h(P_i)<h(P_j)$.
IV:	h is not completely consistent with g.
V:	D is inconsistent with h.
anomaly:	the sufficient conditions are impractical.
exists:	the necessary conditions are too loose.
*:	multiple subproblem lists and only local dominance tests are used.

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